KILLING VECTORS IN HIGHER DIMENSIONAL SPACETIMES WITH CONSTANT SCALAR CURVATURE INVARIANTS

DAVID MCNUTT, NICOS PELAVAS, ALAN COLEY

Abstract. We study the existence of a non-spacelike isometry, ζ, in higher dimensional Kundt spacetimes with constant scalar curvature invariants (CSI). We present the particular forms for the null or timelike Killing vectors and a set of constraints for the metric functions in each case. Within the class of N dimensional CSI Kundt spacetimes, admitting a non-spacelike isometry, we determine which of these can admit a covariantly constant null vector that also satisfy $\zeta_{[a;b]} = 0$.

Introduction

An N dimensional differentiable manifold of Lorentzian signature for which all polynomial scalar curvature invariants constructed from the Riemann tensor and its covariants derivatives are constant is called a CSI spacetime. There are many examples of CSI spacetimes in general relativity and other gravity theories. The higher dimensional pp-wave spacetimes, which are exact solutions of supergravity and string theory of Ricci type N, have vanishing polynomial scalar curvature invariants. We call such spacetimes VSI and note that the set of all VSI spacetimes is a subset of the set of all CSI spacetimes.

There are further subdivisions in the set of all CSI spacetimes depending on distinguishing properties of the spacetimes:

- CSI_R - The set of all reducible CSI spacetimes that can be built from VSI and H by (i) warped products (ii) fibered products, and (iii) tensor sums.
- CSI_F - All spacetimes for which there exists a frame with a null vector ℓ such that all components of the Riemann tensor and its covariants derivatives in this frame have the property that (i) all positive boost weight components (with respect to ℓ) are zero and (ii) all zero boost weight components are constant.
- CSI_K - Those CSI spacetimes that belong to the (higher dimensional) Kundt class; the so-called Kundt CSI spacetimes.

For a Riemannian manifold every CSI spacetime is homogeneous; this is not true for Lorentzian manifolds. However, for every CSI spacetime with particular constant invariants there is a homogeneous spacetime (not necessarily unique) with precisely the same constant invariants. This suggests that CSI spacetimes can be constructed from H and VSI (e.g., CSI_R). In particular, the relationship between CSI_R, CSI_F, CSI_K and especially with $CSI \backslash H$ were studied in arbitrary dimensions [6] (and considered in more detail in the four dimensional case). We note that by construction CSI_R is at least of Weyl type II (i.e., of type II, III, N or O [9]), and by definition CSI_F and CSI_K are at least of Weyl type II (more

Date: October 30, 2012.

precisely, at least of Riemann type II). In four dimensions CSI_R, CSI_F and CSI_K are closely related and that if a spacetime was CSI then it is either homogeneous or belong to the Kundt CSI spacetimes. ([7] [8], [6]). It is conjectured that in higher dimensions this holds true as well.

It was shown in [5] that more generally all Ricci type N VSI spacetimes and some Ricci type III VSI spacetimes (assuming appropriate sources) are solutions to type IIB supergravity; it was argued that these are also solutions to other supergravity theories as well. There are many supergravity CSI spacetimes [3]. The CSI spacetime $AdS_d \times S^{D-d}$ for fixed D is a solution to supergravity as it preserves the maximal number of superymmetries. There are other CSI solutions which admit supersymmetries: the AdS gyratons ([13], [14]) or chiral null models [12]. However, the full set of CSI spacetimes which admit supersymmetries has not been studied in detail.

In [5] it was noted that a particular solution of type IIB supergravity admit a non-spacelike isometry in order to admit a supersymmetry. Furthermore in [5] it was proven that the only VSI spacetimes which admit a null or timelike Killing vector are those which already admit a covariantly constant null vector $\ell = \frac{\partial}{\partial v}$. This implies the set of VSI spacetimes which satisfy type IIB supergravity belong to the subset of VSI Kundt spacetimes [4] with no v dependence. We intend to generalize this result to the set of CSI Kundt spacetimes [4]. That is, we shall determine the set of CSI Kundt spacetimes admitting a null or timelike Killing vector.

Kundt Spacetimes. In higher dimensions it was shown that ℓ is geodesic, non-expanding, shear-free and non-twisting in VSI spacetimes [1]; its covariant derivative takes the form:

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{1i}\ell_a m^i{}_b + L_{i1} m^i{}_a \ell_b. \tag{1}$$

For locally homogeneous spacetimes, in general there exists a null frame in which the L_{ij} are constants. We therefore anticipate that in CSI spacetimes that are not locally homogeneous $L_{ij} = 0$. For higher dimensional CSI spacetimes with $L_{ij} = 0$, the Ricci and Bianchi identities appear to be identically satisfied [11]. A higher dimensional spacetime admitting a null vector ℓ which is geodesic, non-expanding, shear-free and non-twisting, will be denoted as a higher dimensional Kundt spacetime.

It was shown in [6] that there exists a local coordinate system (u, v, x^i) such that

$$ds^2 = 2du\left(dv + H(v,u,x^k)du + W_i(v,u,x^k)dx^i\right) + \tilde{g}_{ij}(u,x^k)dx^i dx^j \tag{2}$$

and the only coordinate transformations preserve the Kundt form [6] are the following:

$$(v',u',x'^i) = (v,u,f^e(x^g)),\ \ with\ J^e{}_f = \tfrac{\partial f^e}{\partial x^f}$$
$$H' = H,\ W'_e = W_f (J^{-1})^f{}_e,\ g'_{ef} = g_{gh}(J^{-1})^g{}_e (J^{-1})^h{}_f, \tag{3}$$

$$(v',u',x'^i) = (v + h(u,x^e), u, x^i)$$
$$H' = H - h_{,u},\ W'_e = W_e - h_{,e},\ g'_{ef} = g_{ef}, \tag{4}$$

$$(v',u',x'^i) = (v/g_{,u}, g(u), x^i)$$
$$H' = \tfrac{1}{g_{,u}^2}(H + v\tfrac{g_{,uu}}{g_{,u}}),\ W'_e = \tfrac{1}{g_{,u}} W_e,\ g'_{ef} = g_{ef}. \tag{5}$$

Furthermore it was shown that for all $CSI \subset CSI_F \bigcap CSI_K$, there always exists (locally) a coordinate transformation $(v',u',x'^i) = (v,u,f^i(u;x^k))$ preserving the form of the Kundt metric, and such that

$$\tilde{g}_{ij} \equiv \tilde{g}'_{kl}\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j}, \quad \tilde{g}'_{ij,u'} = 0$$

where $dS_H^2 = \tilde{g}'_{ij} dx'^i dx'^j$ is a locally homogeneous space.

CSI_0 **Kundt spacetimes.** For a particular coframe,

$$n = dv + H(u,v,x^3)du + w_e(u,v,x^e)dx^e,\ \ell = du,\ m^i = m^i{}_e dx^e$$

with $m^i_e m_{if} = g_{ef}$, the non-zero frame connection components are:

$$\Gamma_{21i} = \tfrac{D_1 W_i}{2},\ \Gamma_{212} = D_1 H,\ \Gamma_{2i2} = D_i H - D_2 W_i, \tag{6}$$
$$\Gamma_{i12} = \tfrac{D_1 W_i}{2},\ \Gamma_{i21} = \tfrac{D_1 W_i}{2},\ \Gamma_{i2j} = \tfrac{A_{ij}}{2},\ \Gamma_{ij2} = \tfrac{A_{ij}}{2}, \tag{7}$$
$$\Gamma_{ijk} = -\tfrac{1}{2}\left(D_{ijk} + D_{jki} - D_{kij}\right) \quad . \tag{8}$$

Where the tensors involved are written in terms of m_{ie} and it's inverse $D_{ijk} = 2m_{ie,f}m_{[j}^{\ e}m_{k]}^{\ f}$, and $A_{ij} = D_{[j}W_{i]} - D_{kji}W^k = 2W_{[i;j]}$.

The linearly independent components of the Riemann tensor with boost weight 1 and 0 may be written as:

$$\begin{aligned}
R_{121i} &= -\frac{1}{2}W_{i,vv} \\
R_{1212} &= -H_{,vv} + \frac{1}{4}\left(W_{i,v}\right)\left(W^{i,v}\right), \\
R_{12ij} &= W_{[i}W_{j],vv} + W_{[i;j],v}, \\
R_{1i2j} &= \frac{1}{2}\left[-W_j W_{i,vv} + W_{i;j,v} - \frac{1}{2}\left(W_{i,v}\right)\left(W_{j,v}\right)\right], \\
R_{ij\hat{i}\hat{j}} &= \tilde{R}_{ij\hat{i}\hat{j}}.
\end{aligned}$$

The spacetime will be CSI_0 if there exists a frame $\{\ell, n, m^i\}$, a constant σ, anti-symmetric matrix $\mathsf{a}_{\hat{i}\hat{j}}$, and symmetric matrix $\mathsf{s}_{\hat{i}\hat{j}}$ such that:

$$W_{\hat{i},vv} = 0, \tag{9}$$

$$H_{,vv} - \frac{1}{4}\left(W_{\hat{i},v}\right)\left(W^{\hat{i},v}\right) = \sigma, \tag{10}$$

$$W_{[\hat{i};\hat{j}],v} = \mathsf{a}_{\hat{i}\hat{j}}, \tag{11}$$

$$W_{(\hat{i};\hat{j}),v} - \frac{1}{2}\left(W_{\hat{i},v}\right)\left(W_{\hat{j},v}\right) = \mathsf{s}_{\hat{i}\hat{j}}, \tag{12}$$

and the components $\tilde{R}_{ij\hat{i}\hat{j}}$ are all constants (i.e., dS_H^2 is curvature homogeneous). We note that (9) and (10) imply that the metric functions take the following form:

$$W_i(v,u,x^e) = vW_i^{(1)}(u,x^e) + W_i^{(0)}(u,x^e), \tag{13}$$

$$H(v,u,x^e) = \frac{v^2}{8}\left[4\sigma + (W_i^{(1)})(W^{(1)i})\right] + vH^{(1)}(u,x^e) + H^{(0)}(u,x^e). \tag{14}$$

The Killing Equations

Let $\zeta = \zeta_1 n + \zeta_2 \ell + \zeta_i m^i$ be a Killing vector field in a CSI Kundt spacetime; it satisfies the Killing equations for $a, b \in [1, N]$

$$\zeta_{a,b} + \zeta_{b,a} - 2\Gamma^c{}_{(ab)}\zeta_c = 0.$$

To simplify the analysis of these equations, we choose new coordinates where one of the Killing vectors of the transverse space, Y, has been rectified so that locally it behaves as a translation; i.e., $Y = A\frac{\partial}{\partial x^3}$. In this coordinate system g_{33} will be constant, and so it is possible to pick a coframe with an upper-triangular matrix $m^i{}_e$ and $m^3{}_3$ constant [15]. This choice of coframe causes Γ_{3ij} and $\Gamma_{3(ij)}$ $\forall i, j \in [3, N]$ to vanish. Rotating the frame so that the spatial component of ζ is locally aligned with m^3, ζ takes the form $\zeta = \zeta_1 n + \zeta_2 \ell + \zeta_3 m^3$

The components ζ_1 and ζ_3 may be partially integrated from the equations with indices (11), (13), ($3i$):

$$\zeta_1 = \zeta_1(u,x^3),\ \zeta_3 = -D_3(\zeta_1)v + \zeta_3^{(0)}(u,x^e), \tag{15}$$

where $\zeta_3^{(0)}$ satisfies the following differential equations from ($3i$):

$$D_i\zeta_3^{(0)} + W_i^{(0)}D_3(\zeta_1) = 0,\ D_iD_3\zeta_1 - W_i^{(1)}D_3\zeta_1 = 0 \tag{16}$$

The tensors $\Gamma_{2i2} = D_iH - D_2W_i$ and $A_{mn} = D_{[n}W_{m]}$ may be expanded into orders of v:

$$\begin{aligned}
\Gamma_{2i2} &= \overbrace{(D_i\sigma^* - \sigma^* W_i^{(1)})}^{\Gamma_i^{(2)}}\frac{v^2}{8} + \overbrace{(D_i H^{(1)} - \frac{1}{4}W_i^{(0)}\sigma^* - D_2 W_i^{(1)})}^{\Gamma_i^{(1)}}v \\
&+ \overbrace{D_i H^{(0)} - W_i^{(0)}H^{(1)} - D_2 W_i^{(0)} + H^{(0)}W_i^{(1)}}^{\Gamma_i^{(0)}}, \qquad (17) \\
A_{ij} &= \overbrace{2D_{[j}W_{i]}^{(1)}}^{A_{ij}^{(1)}}v + \overbrace{2D_{[j}W_{i]}^{(0)} - 2W_{[j}^{(0)}W_{i]}^{(1)}}^{A_{ij}^{(0)}}.
\end{aligned}$$

Here $\sigma^* \equiv 4\sigma + W_i^{(1)}W^{(1)i}$ and the metric functions H and $W_i = m_i{}^e w_e$ are of the form (13) and (14). Substituting these into the equation with indices (21) yields ζ_2 in orders of v:

$$(18)\zeta_2 = \overbrace{(\frac{\sigma^*\zeta_1}{4} - W_3^{(1)}D_3(\zeta_1))}^{\zeta_2^{(2)}}\frac{v^2}{2} + \overbrace{(W_3^{(1)}\zeta_3^{(0)} - D_2\zeta_1 + H^{(1)}\zeta_1)}^{\zeta_2^{(1)}}v + \zeta_2^{(0)}(u, x^e).$$

Our primary interest are those CSI spacetimes which do not admit covariantly constant null vectors, since the existence of Killing vectors in $CCNV$ spacetimes was considered in [10]. The analysis will be restricted to non-spacelike Killing vectors, $|\zeta| \leq 0$. Using the definition of the vector components given above the magnitude is expanded into orders of v:

$$\frac{-\sigma^*}{4}(\zeta_1)^2 + W_3^{(1)}D_3(\zeta_1)\zeta_1 + (D_3(\zeta_1))^2 \leq 0 \qquad (19)$$

$$\zeta_1(W_3^{(1)}\zeta_3^{(0)} - D_2\zeta_1 + H^{(1)}\zeta_1) + D_3(\zeta_1)\zeta_3^{(0)} = 0 \qquad (20)$$

$$(\zeta_3^{(0)})^2 - 2\zeta_1\zeta_2^{(0)} \leq 0. \qquad (21)$$

The remaining Killing equations, with indices 22, 23 and $2n$ are now expanded into orders of v, giving the following set of equations:

$$\Gamma_3^{(2)}D_3(\zeta_1) = 0, \qquad (22)$$

$$D_2\zeta_2^{(2)} + \tfrac{1}{4}\sigma^*\zeta_2^{(1)} - H^{(1)}\zeta_2^{(2)} - \tfrac{1}{4}\Gamma_3^{(1)}D_3(\zeta_1) + \tfrac{1}{4}\Gamma_3^{(2)}\zeta_3^{(0)} = 0, \qquad (23)$$

$$D_2\zeta_2^{(1)} + \tfrac{1}{4}\sigma^*\zeta_2^{(0)} - H^{(0)}\zeta_2^{(2)} - \Gamma_3^{(0)}D_3(\zeta_1) + \Gamma_3^{(1)}\zeta_3^{(0)} = 0, \qquad (24)$$

$$D_2\zeta_2^{(0)} - H^{(0)}\zeta_2^{(1)} + H^{(1)}\zeta_2^{(0)} + \Gamma_3^{(0)}\zeta_3^{(0)} = 0, \qquad (25)$$

$$\tfrac{1}{4}\sigma^*D_3(\zeta_1) + D_3\zeta_2^{(2)} - W_3^{(1)}\zeta_2^{(2)} - \tfrac{1}{4}\Gamma_3^{(2)}\zeta_1 = 0, \qquad (26)$$

$$D_2D_3(\zeta_1) - H^{(1)}D_3(\zeta_1) - D_3\zeta_2^{(1)} + W_3^{(0)}\zeta_2^{(2)} + \Gamma_3^{(1)}\zeta_1 = 0, \qquad (27)$$

$$D_2\zeta_3^{(0)} + H^{(0)}D_3(\zeta_1) + D_3\zeta_2^{(0)} - W_3^{(0)}\zeta_2^{(1)} - \Gamma_3^{(0)}\zeta_1 + W_3^{(1)}\zeta_2^{(0)} = 0, \qquad (28)$$

$$D_n\zeta_2^{(2)} - W_n^{(1)}\zeta_2^{(2)} - \tfrac{1}{4}\Gamma_n^{(2)}\zeta_1 + A_{3n}^{(1)}D_3(\zeta_1) = 0, \tag{29}$$
$$D_n\zeta_2^{(1)} - W_n^{(0)}\zeta_2^{(2)} - \Gamma_n^{(1)}\zeta_1 - A_{3n}^{(1)}\zeta_3^{(0)} + A_{3n}^{(0)}D_3(\zeta_1) = 0, \tag{30}$$
$$D_n\zeta_2^{(0)} - W_n^{(0)}\zeta_2^{(1)} - \Gamma_n^{(0)}\zeta_1 + W_n^{(1)}\zeta_2^{(0)} - A_{3n}^{(0)}\zeta_3^{(0)} = 0. \tag{31}$$

The analysis splits into subcases arising from (22) where either $D_3(\zeta_1)$ or $\Gamma_3^{(2)}$ are assumed to vanish seperately.

Implications of $\zeta_{[a;b]} = 0$

Before each case is analyzed it will be beneficial to examine the anti symmetrization of $\zeta_{a;b} = 0$ to determine the set of CSI spacetimes admitting a covariantly constant non-spacelike vector. Non-spacelike Killing vectors in $CCNV$ CSI spacetimes has already been studied in [10] as such if a CSI spacetime is shown to be $CCNV$ it may be disregarded in the current analysis. Conversely it is of interest to determine when a CSI spacetime admits a Killing vector but cannot admit a covariantly constant vector.

Using the form of ζ given above, the vanishing of $\zeta_{[a;b]}$, yields the following equations:

$$D_2\zeta_1 - D_1\zeta_2 - \Gamma^1{}_{12}\zeta_1 = 0, \tag{32}$$
$$D_3\zeta_1 - D_1\zeta_3 - 2\Gamma^1{}_{[13]}\zeta_1 = 0, \tag{33}$$
$$2\Gamma^1{}_{[1n]}\zeta_1 = 0, \tag{34}$$
$$D_3\zeta_2 - D_2\zeta_3 + \Gamma^1{}_{32}\zeta_1 = 0, \tag{35}$$
$$D_n\zeta_2 + \Gamma^1{}_{n2}\zeta_1 = 0, \tag{36}$$
$$D_n\zeta_3 - 2\Gamma^1{}_{[3n]}\zeta_1 = 0, \tag{37}$$
$$\Gamma^1{}_{[nm]}\zeta_1 = 0. \tag{38}$$

Assuming $\zeta_1 \neq 0$ and expanding (34) implies $\Gamma^1{}_{[1n]} = W_n^{(1)} = 0$. Similarly $2\Gamma^1{}_{[13]} = W_3^{(1)}$ and so equation (33) gives $W_3^{(1)} = 2D_3 ln(\zeta_1)$. Equation (38) implies that A_{nm} must vanish. Using (17) we may summarize these observations as

Lemma 0.1. *For those spacetimes admitting a vector ζ such that $\zeta_{(a;b)} = 0$ and $\zeta_{[a;b]} = 0$ it is necessary that the metric functions W_i satisfy the following:*

$$W_3^{(1)} = 2D_3 ln(\zeta_1),\ W_n^{(1)} = 0,$$
$$A_{nm} = 2D_{[m}W_{n]}^{(0)} = 0.$$

The remaining equations are:

$$D_2\zeta_1 - D_1\zeta_2 - \Gamma_{212}\zeta_1 = 0, \tag{39}$$
$$D_3\zeta_2 - D_2\zeta_3 + \Gamma_{232}\zeta_1 = 0, \tag{40}$$
$$D_n\zeta_2 + \Gamma_{2n2}\zeta_1 = 0, \tag{41}$$
$$D_n\zeta_3 - A_{3n}\zeta_1 = 0. \tag{42}$$

These will be studied once the analysis of the Killing equations has been completed.

Case 1: $D_3(\zeta_1) = 0$

Setting $D_3(\zeta_1)$ equal to zero we obtain

$$\zeta_1 = \zeta_1^{[0]}(u), \ \zeta_3 = \zeta_3^{(0)}(u) \tag{43}$$

$$\zeta_2 = \overbrace{(\frac{\sigma^*\zeta_1}{4})}^{\zeta_2^{(2)}} \frac{v^2}{2} + \overbrace{(W_3^{(1)}\zeta_3 - D_2\zeta_1 + H^{(1)}\zeta_1)}^{\zeta_2^{(1)}} v + \zeta_2^{(0)}(u, x^e). \tag{44}$$

The non-spacelike conditions are now

$$-\sigma^*(\zeta_1)^2 \leq 0, \ \zeta_1(W_3^{(1)}\zeta_3 - D_2\zeta_1 + H^{(1)}\zeta_1) = 0, \ (\zeta_3)^2 - \zeta_1\zeta_2^{(0)} \leq 0 \tag{45}$$

so either ζ_1 vanishes and ζ is a null Killing vector or $\zeta_1 \neq 0$ and $\sigma^* \geq 0$.

Case 1.1 : $\zeta_1 = 0$. If ζ_1 is allowed to vanish, the remaining non-spacelike conditions imply that $\zeta_3 = 0$ and so the Killing vector is of the form $\zeta = \zeta_2\ell$. In light of the special form of ζ_2 it must be a function of only u and the spatial coordinates x^e. The remaining Killing equations are

$$\sigma^*\zeta_2^{(0)} = 0 \tag{46}$$

$$D_2\zeta_2^{(0)} + H^{(1)}\zeta_2^{(0)} = 0 \tag{47}$$

$$D_3\zeta_2^{(0)} + W_3^{(1)}\zeta_2^{(0)} = 0 \tag{48}$$

$$D_n\zeta_2^{(0)} + W_n^{(1)}\zeta_2^{(0)} = 0. \tag{49}$$

The vanishing of σ^* in the first term (46) implies $W_i^{(1)}W^{(1)i} = -4\sigma$ where $W_i^{(1)} = m_i{}^e w_e^{(1)}$ and hence

$$W_i^{(1)}W^{(1)i} = g^{ef}W_e^{(1)}W_f^{(1)} = -4\sigma. \tag{50}$$

Since the transverse metric is Riemannian, it is positive-definite and restricts the value of σ to be less than or equal to zero.

Case 1.1.1: If $\sigma = 0$, this implies $W_e^{(1)} = 0$ for all $e \in [3, N]$. The vector component ζ_2 will be a function of u only and the remaining equation (47) determines the metric function

$$H^{(1)}(u) = -D_2 ln(\zeta_2). \tag{51}$$

One may always make a coordinate transform of the form (5) to set $H^{(1)} = 0$, so the metric is independent of the null coordinate v and $\zeta = \ell = \frac{\partial}{\partial v}$, implying that ζ is a covariantly constant null vector.

Case 1.1.2: If $\sigma < 0$, one may solve for the metric functions $W_i^{(1)}$ and $H^{(1)}$ in terms of $\zeta(u, x^e)$:

$$H^{(1)}(u, x^e) = -D_2 ln(\zeta_2), \ W_i^{(1)}(u, x^e) = -D_i ln(\zeta_2).$$

These CSI spacetimes do not admit a covariantly constant vector. To see this, assume $\sigma < 0$ and consider equations (39) - (41); the first two are automatically

satsified while the last implies that ζ_2 is a function of u only. This forces the $W_i^{(1)}$ to all vanish, leading to the contradiction: $0 = \sigma^* = \sigma < 0$, hence these spacetimes do not admit a $CCNV$.

Given a null vector of the form, $\zeta = \zeta(u, x^e)\ell$, it will be a Killing vector for the CSI spacetime with a locally homogeneous transverse space and metric functions:

$$H = -(ln\zeta)_{,u} v + H^{(0)}(u, x^e),\ W_e = -(ln\zeta)_{,e} v + W_e^{(0)}(u, x^f). \tag{52}$$

The vanishing of the function σ^* leads to one last condition for the CSI spacetime. Since $W_e^{(1)} = -(ln\zeta)_{,e}$, the only constraint on the function ζ arises from (12).

$$\sum_{i=3}^{N} [D_i ln(\zeta_2)]^2 = -4\sigma,\ \sigma < 0 \tag{53}$$

$$g^{ef} W_e^{(1)} W_f^{(1)} = -4\sigma.$$

The left-hand-side must be positive, and so it is necessary that $\sigma = R_{1212}$ is a negative real number.

Case 1.2 : The remaining conditions from $|\zeta| \leq 0$ are

$$D_2\zeta_1 - H^{(1)}\zeta_1 = W_3^{(1)}\zeta_3 \tag{54}$$

$$(\zeta_3)^2 \leq \zeta_1\zeta_2^{(0)} \quad . \tag{55}$$

Expanding ζ_2 and $\Gamma_3^{(1)}$, we find that the $O(v^2)$ terms, (26) and (29) are automatically satisfied, while (23) and using (54) yield the following differential equation for $\sigma^*(u, x^e)$:

$$\zeta_1 D_2\sigma^* + \zeta_3 D_3\sigma^* = 0. \tag{56}$$

Using a coordinate transformation of the form (5), coordinates are chosen so that $\zeta_1 = 1$ and the non-spacelike condition (54) determines a part of H

$$H^{(1)} = -W_3^{(1)}\zeta_3.$$

We can apply another coordinate transform of type (4) to eliminate $H^{(0)}$ as well. In this coordinate system the Killing equations are:

$$\sigma^*(\zeta_2^{(0)} - \zeta_3 W_3^{(0)}) = 0, \tag{57}$$

$$D_2\zeta_2^{(0)} + \zeta_3 D_3\zeta_2^{(0)} + \zeta_3 D_2\zeta_3 = 0, \tag{58}$$

$$D_2 W_i^{(1)} + \zeta_3 D_3 W_i^{(1)} = 0, \tag{59}$$

$$D_2 W_3^{(0)} - \zeta_3 W_3^{(0)} W_3^{(1)} = -D_2\zeta_3 - D_3\zeta_2^{(0)} - W_3^{(1)}\zeta_2^{(0)}, \tag{60}$$

$$D_2 W_n^{(0)} + \zeta_3 D_3 W_n^{(0)} = (W_3^{(0)} W_n^{(1)} + D_n W_3^{(0)})\zeta_3 - W_n^{(1)}\zeta_2^{(0)} - D_n\zeta_2^{(0)}, \tag{61}$$

If ζ_3 is non-zero, equation (58) simplifies the differential equation for $W_3^{(0)}$ in (60). Thus two subcases must be considered in which ζ vanishes or not.

Case 1.2.1: Setting ζ_3 equal to zero causes $H^{(1)}$ to vanish while (58) and (59) imply

$$D_2 W_i^{(1)} = D_2 \zeta_2^{(0)} = 0. \tag{62}$$

The remaining equations give constraints for the remaining metric functions:

$$\sigma^*(\zeta_2^{(0)}) = 0, \tag{63}$$
$$D_2 W_3^{(0)} = -D_3 \zeta_2^{(0)} - W_3^{(1)} \zeta_2^{(0)}, \tag{64}$$
$$D_2 W_n^{(0)} = -W_n^{(1)} \zeta_2^{(0)} - D_n \zeta_2^{(0)}. \tag{65}$$

Thus there are two minor subcases to consider arising from (63).

Case 1.2.1a. Assuming $\sigma^* \neq 0$, $\zeta_2^{(0)}$ vanishes and the set of spacetimes with metric functions:

$$H(v, x^e) = \sigma^* \frac{v^2}{8},\ W_i(v, x^e) = W_i^{(1)}(x^e) v + W_i^{(0)}(x^e) \tag{66}$$

are $CCNV$ spacetimes with $\frac{\partial}{\partial u}$ as a covariantly constant null vector admitting a Killing vector of the form:

$$\zeta = n + \frac{\sigma^* v^2}{8} \ell.$$

If we suppose ζ is a covariantly constant vector; Lemma (0.1) and equations (40) and (41) force the metric functions $W_i^{(1)}$ and $W_n^{(0)}$ to vanish. However a contradiction arises from (39) as it requires $\sigma^* = 0$ but we have assumed that $\sigma^* \neq 0$ and so the above spacetime cannot admit a covariantly constant vector.

Case 1.2.1b. For the other subcase, σ^* is equal to zero, and the positive-definite signature of the transverse metric restricts $\sigma \leq 0$. For arbitrary $\zeta_2^{(0)}(x^e)$, and any choice of $W_i^{(1)}(x^e)$ satisfying (50) with $\sigma = R_{1212} \leq 0$, the CSI Kundt spacetime with a locally homogeneous transverse space and metric functions:

$$H = 0,\ W_i(u, v, x^e) = W_i^{(1)} v - (D_i \zeta_2^{(0)} + W_i^{(1)} \zeta_2^{(0)}) u + w_i(x^e) \tag{67}$$

admit a Killing vector of the form:

$$\zeta = n + \zeta_2^{(0)} \ell$$

To preserve the non-spacelike requirement $\zeta_2^{(0)}$ must always be greater than or equal to zero. If this killing vector is covariantly constant, $W_i^{(1)} = 0$ and hence $\sigma = 0$, equation (42) implies $A_{ij} = 0$, and the remaining equations (40) and (41) force $\zeta_2^{(0)}$ to be constant. Thus ζ is the sum of the $CCNV$'s ℓ and n.

Case 1.2.2: $\zeta_3 \neq 0$. Divide by ζ_3 in (58) and substitute the result into (60) to simplify the differential equation for $W_3^{(0)}$:

$$D_2 W_3^{(0)} - \zeta_3 W_3^{(0)} W_3^{(1)} = \frac{D_2 \zeta_2^{(0)}}{\zeta_3} - W_3^{(1)} \zeta_2^{(0)} \tag{68}$$

then by multiplying the above by $E(u, x^e) = e^{-[\int W_3^{(1)} \zeta_3 du]}$, integration by parts gives the solution

$$W_3^{(0)} = \frac{\zeta_2^{(0)}}{\zeta_3} + e^{[\int W_3^{(1)} \zeta_3 du]} \int \frac{\zeta_2^{(0)} D_2 \zeta_3}{(\zeta_3)^2 e^{[\int W_3^{(1)} \zeta_3 du]}} du.$$

From (57) there are two minor subcases to consider, depending upon whether σ^* vanishes or not.

Case 1.2.2a : Supposing that σ^* does indeed vanish, the functions $W_3^{(1)}(x^e)$ and $W_n^{(1)}$ must satisfy (50) with $\sigma \leq 0$. For arbitrary $\zeta_3(u)$ and any solution of the following differential equation

$$D_2 \zeta_2^{(0)} + \zeta_3 D_3 \zeta_2^{(0)} = -\zeta_3 D_2 \zeta_3 \tag{69}$$

the Kundt CSI spacetime with a locally homogeneous transverse space and

$$\begin{gathered} H = -W_3^{(1)} \zeta_3 v \\ W_3(u, v, x^e) = W_3^{(1)}(u, x^e) v + \tfrac{\zeta_2^{(0)}}{\zeta_3} + \tfrac{1}{E} \int \tfrac{E \zeta_2^{(0)} D_2 \zeta_3}{(\zeta_3)^2} du, \ E = e^{\int H^{(1)} du} \\ W_n(u, v, x^e) = W_n^{(1)}(u, x^e) v + W_n^{(0)}(u, x^e) \end{gathered} \tag{70}$$

satisfying the following differential equations:

$$D_2 W_i^{(1)} + \zeta_3 D_3 W_i^{(1)} = 0 \tag{71}$$

$$D_2 W_n^{(0)} + \zeta_3 D_3 W_n^{(0)} = \tfrac{\zeta_3 W_n^{(1)}}{E} \int \tfrac{E \zeta_2^{(0)}}{(\zeta_3)^2} du + D_n[\tfrac{\zeta_3}{E} \int \tfrac{E \zeta_2^{(0)} D_2 \zeta_3}{(\zeta_3)^2} du] \tag{72}$$

admits a Killing vector of the form

$$\ell + \zeta_2^{(0)}(u, x^e) n + \zeta_3(u) m^3$$

Requiring ζ to be a $CCNV$, the $W_i^{(1)}$ must vanish, causing $H = 0$ and $\sigma = 0$. This is an example of a $CCNV$ metric, with $\ell = \frac{\partial}{\partial v}$ as the $CCNV$, where ζ will be a second $CCNV$. The additional constraints (39) - (42) imply $A_{ij} = 0$ while the remaining equations lead to two possible subcases for Kundt spacetimes admitting a covariantly constant vector, either $D_n \zeta_2 = 0$ or $D_2 \zeta_3 = 0$. The first case leads to the following form for ζ and the metric functions

$$\begin{gathered} \zeta = n + [-\zeta_3^2] \ell + \zeta_3(u) m^3, \ 3\zeta_3^2 \leq 0 \\ H = 0, \ W_3(u, x^e) = -\zeta_3 + w_3(x^e), \ W_n(x^e) = \int D_n w_3 dx^3 + w_n(x^r). \end{gathered}$$

The non-spacelike condition $3\zeta_3^2 \leq 0$ eliminates the above case, as we've assumed $\zeta_3 \neq 0$ this case is not admissible. In the second case ζ_3 must be constant, scaling x^3 so that $\zeta_3 = 1$,

$$\zeta = n + \zeta_2(x^r)\ell + m^3,\ 1 \leq 2\zeta$$
$$H = 0,\ W_3(x^e) = w_3(x^e),\ W_n(x^e) = \int D_n(w_3)dx^3 - 2D_n(\zeta_2)x^3 + w_n(x^r),$$

The vanishing of $A_{3j} = D_{[j}W_{3]}^{(0)}$ implies that $D_n\zeta_2^{(0)} = 0$ and so $\zeta_2^{(0)}$ must be constant. If ζ is timelike, the constant $\zeta_2^{(0)} > \frac{1}{2}$ while if ζ is null $\zeta_2^{(0)} = \frac{1}{2}$. These spacetimes will automatically be $CCNV$ spacetimes with ℓ as another covariantly constant null vector.

0.1. **Case 1.2.2b:** If σ^* is non-zero, it satisfies the differential equation (56)

$$D_2\sigma^* + \zeta_3 D_3\sigma^* = 0$$

and the identity $\zeta_2^{(0)} = \zeta_3 W_3^{(0)}$ may be derived from (57), which causes (68) to simplify, implying $\zeta_2^{(0)} D_2\zeta_3 = 0$. Letting $\zeta_2^{(0)} = 0$, the differential equation (69) for $\zeta_2^{(0)}$ forces $D_2\zeta_3 = 0$. In either case ζ_3 must be constant and henceforth will be set to one. For any solution $\zeta_2^{(0)}$ to the differential equation

$$D_2\zeta_2^{(0)} + D_3\zeta_2^{(0)} = 0, \tag{73}$$

the vector

$$n + [\frac{\sigma^*}{8}v^2 + \zeta_2^{(0)}]\ell + m^3$$

will be a Killing vector for any CSI Kundt spacetime of the form

$$H = \tfrac{\sigma^*}{8}v^2 - W_3^{(1)}\zeta_3 v, \tag{74}$$
$$W_3(u,v,x^e) = W_3^{(1)}(u,x^e)v + \zeta_2^{(0)},\ W_n(u,v,x^e) = W_n^{(1)}(u,x^e)v + W_n^{(0)}(u,x^e)$$

where the $W_i^{(1)}$ and $W_i^{(0)}$ satisfy the following equations:

$$D_2W_i^{(1)} + D_3W_i^{(1)} = 0, \tag{75}$$
$$D_2W_i^{(0)} + D_3W_i^{(0)} = 0. \tag{76}$$

If ζ is required to be covariantly constant, a contradiction arises from (39) as it requires $\sigma^* = 0$ despite the fact that we have assumed $\sigma^* \neq 0$. Thus there are no $CCNV$ spacetimes of the form (74).

Case 2 : $\Gamma_3^{(2)} = 0$

For the remainder of this case we shall assume $D_3\zeta_1 \neq 0$ to avoid the previous subcases. Supposing $W_3^{(1)} = 0$, this implies that $D_3D_3\zeta_1 = 0$ and $\sigma^* = \sigma$. This causes a contradiction to arise between the Killing equation (26) and the non-spacelike condition (19):

$$2\sigma D_3(\zeta_1) = 0,\ (D_3\zeta_1)^2 \leq \sigma(\zeta_1)^2.$$

The first implies that $\sigma = 0$ as we have assumed $D_3\zeta_1 \neq 0$; however, by the second inequality the vanishing of σ implies $D_3\zeta_1 = 0$ which contradicts our original assumption. Thus $D_3D_3\zeta_1$ is always non-zero, and using this fact we may derive another identity for $\sigma^* = 4\sigma + (W_3^{(1)})^2$ in terms of ζ_1 from the vanishing of $\Gamma_3^{(2)}$:

$$\sigma^* = \frac{D_3\sigma^*}{W_3^{(1)}} = \frac{2W_3^{(1)}D_3W_3^{(1)}}{W_3^{(1)}} = 2D_3D_3(lnD_3\zeta_1). \tag{77}$$

Using a coordinate transform of type (5) with $g(u) = \frac{u}{\sqrt{|\sigma|}}$, we may rescale σ in (10) so that it equals $\sigma = -1, 0, 1$ depending on it's sign. Doing so will scale all of the metric functions and Killing vector components by a constant value, but otherwise will leave them unchanged.

Dropping the primes and substituting (77) into the original identity for σ^* yields another differential equation for $D_3\zeta_1$:

$$D_3D_3ln(D_3\zeta_1) - \frac{1}{2}(D_3ln(D_3\zeta_1))^2 = 2\sigma.$$

Multiplication by $exp(-\frac{1}{2}\int D_3(lnD_3\zeta_1)dx^3) = (D_3\zeta_1)^{-\frac{1}{2}}$ leads to the simpler equation

$$D_3D_3[(D_3\zeta_1)^{-\frac{1}{2}}] = -\sigma(D_3\zeta_1)^{-\frac{1}{2}}. \tag{78}$$

There are three possible solutions to this equation depending on whether σ is positive, negative or zero:

$$\begin{aligned}
\sigma = -1 \quad &: \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1(u)cosh(x^3) + c_2(u)sinh(x^3),\\
\sigma = 0 \quad &: \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1'(u)x^3 + c_2',\\
\sigma = 1 \quad &: \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1''(u)cos(x^3) + c_2''(u)sin(x^3).
\end{aligned}$$

Ignoring these facts for a moment, we recall that the metric functions W_i may be expressed in terms of $\zeta_3^{(0)}$ and ζ_1 using (16) :

$$W_i^{(0)} = \frac{-D_i\zeta_3^{(0)}}{D_3\zeta_1},\ W_i^{(1)} = D_iln(D_3\zeta_1).$$

In this case, it is possible to set all but $W_3^{(1)}$ to zero by making a coordinate transform of type (4) with $h = -\frac{\zeta_3^{(0)}}{D_3\zeta_1}$. In these new coordinates, the metric functions take the form:

$$W_3 = D_3ln(D_3\zeta_1)v,\ W_n = 0. \tag{79}$$

The following coefficient functions of H change in the new coordinate system:

$$H^{(1)} = H'^{(1)} + \frac{\zeta_3^{(0)}\sigma^*}{4D_3\zeta_1},\ H^{(0)} = H'^{(0)} + \frac{\zeta_3^{(0)}H'^{(1)}}{D_3\zeta_1} + D_2\left(\frac{\zeta_3^{(0)}}{D_3\zeta_1}\right)\frac{\sigma^*(\zeta_3^{(0)})^2}{8(D_3\zeta_1)^2},$$

$$H^{(1)} = H'^{(1)} + \frac{\zeta_3^{(0)}\sigma^*}{4D_3\zeta_1},\ H^{(0)} = H'^{(0)} + \frac{\zeta_3^{(0)}H'^{(1)}}{D_3\zeta_1} + D_2\left(\frac{\zeta_3^{(0)}}{D_3\zeta_1}\right)\frac{\sigma^*(\zeta_3^{(0)})^2}{8(D_3\zeta_1)^2},$$

where primed functions denote the functions in the previous coordinate system. As the original $H'^{(1)}$ and $H'^{(0)}$ were arbitrary functions of u and the spatial coordinates, we may ignore the special form the v-coefficients take in this coordinate system and treat them simply as new arbitrary functions. In this coordinate system the tensor A_{3n} given in (17) vanishes, and the connection coefficients Γ_{2i2} are of the form:

$$\Gamma_{2i2} = \overbrace{(D_iH^{(1)} - D_2W_i^{(1)})}^{\Gamma_i^{(1)}}v + \overbrace{D_iH^{(0)} + H^{(0)}W_i^{(1)}}^{\Gamma_i^{(0)}}.$$

This choice of coordinate system simplifies the Killing equations considerably; for example, the other two covector components are now

$$\zeta_2 = \overbrace{(\frac{\sigma^*\zeta_1}{4} - D_3D_3\zeta_1)}^{\zeta_2^{(2)}}\frac{v^2}{2} + \overbrace{(H^{(1)}\zeta_1 - D_2\zeta_1)}^{\zeta_2^{(1)}}v + \zeta_2^{(0)}(u, x^e),$$
$$\zeta_3 = -D_3(\zeta_1)v.$$

Taking the magnitude of the vector and invoking the non-spacelike conditions yield

$$D_3D_3ln[(D_3\zeta_1)^{-\frac{1}{2}}] + D_3(ln(D_3\zeta_1))D_3ln(\zeta_1) + (D_3ln(\zeta_1))^2 \leq 0, \tag{80}$$
$$\zeta_1(H^{(1)}\zeta_1 - D_2\zeta_1) = 0, \tag{81}$$
$$\zeta_1\zeta_2^{(0)} \geq 0. \tag{82}$$

Thus $\zeta_2^{(1)}$ must vanish and we may solve for $H^{(1)}$ in terms of ζ_1,

$$H^{(1)} = D_2ln(\zeta_1).$$

Further constraints on H involving $H^{(0)}$ may be found by taking those Killing equations involving the spatial derivatives of $\zeta_2^{(0)}$; i.e., (28) and (31) and considering integrability conditions. In this coordinate system (28) and (31) are

$$D_3\zeta_2^{(0)} + H^{(0)}D_3\zeta_1 - \zeta_1D_3H^{(0)} - \zeta_1H^{(0)}D_3ln(D_3\zeta_1) + \zeta_2^{(0)}D_3ln(D_3\zeta_1) = 0$$
$$D_n\zeta_2^{(0)} - \zeta_1D_nH^{(0)} = 0$$

We note that the commutator applied to any function independent of v vanishes (i.e. $[D_3, D_n]f(u, x^e) = 0$); thus differentiating the first equation by D_n and the latter by D_3 and subtracting the result gives the following constraint

$$2D_n(H^{(0)})D_3\zeta_1 = 0.$$

Hence $H^{(0)}$ and $\zeta_2^{(0)}$ are actually functions of u and the spatial coordinate x^3.

In light of this fact the Killing equations (29) - (31) are automatically satisfied. Similarly, equation (26) may be ignored as it gives the identity $\sigma^* = 2D_3D_3ln(D_3\zeta_1)$, which arose from the vanishing of $\Gamma_3^{(2)}$. The remaining Killing equations are now:

$$D_2\sigma^* = 4D_2\left(\frac{D_3D_3\zeta_1}{\zeta_1}\right) - \frac{1}{2}D_2[(D_3ln(\zeta_1))^2], \tag{83}$$

$$D_3H^{(0)} = \frac{\sigma^*}{4D_3\zeta_1}(\zeta_2^{(0)} - H^{(0)}\zeta_1), \tag{84}$$

$$D_2\zeta_2^{(0)} = -\zeta_2^{(0)}D_2ln(\zeta_1), \tag{85}$$

$$2D_2D_3ln(\zeta_1) = D_2D_3ln(D_3\zeta_1), \tag{86}$$

$$D_3(\zeta_2^{(0)}D_3\zeta_1) = \zeta_1^2D_3[H^{(0)}D_3ln(\zeta_1)]. \tag{87}$$

Differentiating (86) and using the fact that $[D_3, D_2]f(u, x^e) = 0$, one finds the following expression for $D_2\sigma^* = 2D_2D_3D_3ln(D_3\zeta_1)$:

$$D_2\sigma^* = 4D_2\left[\frac{D_3D_3\zeta_1}{\zeta_1} - \left(\frac{D_3\zeta_1}{\zeta_1}\right)^2\right].$$

Subtracting this from (83) yields the following constraint

$$D_2\left(D_3ln(\zeta_1)\right)^2 = 0$$

implying that ζ_1 must take the form:

$$\zeta_1 = e^{A(x^3)}e^{B(u)}. \tag{88}$$

Apply a coordinate transform of type (5) with $g = \int e^{-B(u)}du$ will remove the u dependence from ζ_1. Rewriting (85) in terms of ${\zeta'}_2^{(0)} = \zeta_2^{(0)}e^B$, it is easily shown that this implies $D_2{\zeta'}_2^{(0)} = 0$. Denoting $\zeta_1' = e^{A(x^3)}$ the Killing vector $\zeta = e^Ae^Bn + \zeta_2\ell + \zeta_3m^3$ becomes:

$$\zeta = \zeta_1'n' + \left[\left(\frac{\sigma^*\zeta_1'}{4} - D_3D_3(\zeta_1')\right)\frac{v'^2}{2} + {\zeta'}_2^{(0)}(x^3)\right]\ell' + [-D_3(\zeta_1')v']m^3.$$

In the remaining Killing equations, (84) and (87), the function $H^{(0)}$ in the new coordinate system becomes ${H'}^{(0)} = e^{2B}H^{(0)}$ and so we may remove e^B entirely from these two equations.

Dropping the primes and combining (84) with (87) yields the following algebraic equation for $H^{(0)}$:

$$H^{(0)}\left(D_3D_3ln(\zeta_1) - \frac{\sigma^*}{4}\right) = \frac{D_3(D_3(\zeta_1)\zeta_2^{(0)})}{\zeta_1^2} - \frac{\sigma^*\zeta_2^{(0)}}{4\zeta_1}$$

The coefficient of $H^{(0)}$ cannot vanish, as the non-spacelike condition (80) would imply

$$2(D_3ln(\zeta_1))^2 \leq 0.$$

It is assumed that $D_3\zeta_1 \neq 0$ so the above constraint is impossible. Simplifying the above expression $H^{(0)}$ may be written as

$$H^{(0)} = \frac{D_3(D_3(\zeta_1)\zeta_2^{(0)}) + D_3D_3ln((D_3\zeta_1)^{-\frac{1}{2}})\zeta_2^{(0)}\zeta_1}{\zeta_1^2 D_3D_3ln(\zeta_1(D_3\zeta_1)^{-\frac{1}{2}})} \tag{89}$$

Having exhausted the Killing equations, we look to the remaining non-spacelike conditions (80) and (82).

Case 2.1: Null Killing Vectors. If ζ is required to be null $\zeta_2^{(0)}$ must be zero, forcing $H^{(0)}$ to vanish as well. Using (80) and (78) we find the following expression

$$D_3(A) = D_3ln[(D_3\zeta_1)^{-\frac{1}{2}}] \pm \sqrt{2[D_3ln((D_3\zeta_1)^{-\frac{1}{2}})]^2 + \sigma}. \tag{90}$$

Combining this with the solution to (78) for a particular $\sigma = -1, 0, 1$:

$$\sigma = -1 \quad : \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1cosh(x^3) + c_2sinh(x^3) \tag{91}$$

$$\sigma = 0 \quad : \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1x^3 + c_2 \tag{92}$$

$$\sigma = 1 \quad : \quad (D_3\zeta_1)^{-\frac{1}{2}} = c_1cos(x^3) + c_2sin(x^3) \tag{93}$$

we may algebraically solve for ζ_1 by noting that $D_3\zeta_1 = D_3(A)e^A = D_3(A)\zeta_1$:

$$\sigma = -1 : \zeta_1 = \frac{(c_1cosh(x^3)+c_2sinh(x^3))^{-1}}{c_1sinh(x^3)+c_2cosh(x^3)\pm\sqrt{c_1^2+c_2^2+(c_1sinh(x^3)+c_2cosh(x^3))^2}} \tag{94}$$

$$\sigma = 0 : \zeta_1 = \frac{1}{c_1(1\pm\sqrt{2})(c_1x^3+c_2)} \tag{95}$$

$$\sigma = 1 : \zeta_1 = \frac{(c_1cos(x^3)+c_2sin(x^3))^{-1}}{-c_1sin(x^3)+c_2cos(x^3)\pm\sqrt{c_1^2+c_2^2+(-c_1sin(x^3)+c_2cos(x^3))^2}} \tag{96}$$

Supposing that ζ is covariantly constant, the constraint in Lemma (0.1) on $W_3^{(1)}$ along with the identity (16) yields

$$D_3ln(\zeta_1) = -D_3ln[(D_3\zeta_1)^{\frac{1}{2}}]. \tag{97}$$

Since $ln(\zeta_1) = A$, the above simplifies (90) in the null case, giving

$$2D_3ln[(D_3\zeta_1)^{-\frac{1}{2}}] \pm \sqrt{2[D_3ln((D_3\zeta_1)^{-\frac{1}{2}})]^2 + \sigma} = 0$$

Multiplying both roots together the result must vanish

$$2[D_3ln[(D_3\zeta_1)^{-\frac{1}{2}}]]^2 - \sigma = 0 \tag{98}$$

Substituting the three posibilities of $(\zeta_1)^{\frac{1}{2}}$ gives the constraint:

$$3[c_1^2 - c_2^2]sinh^2(x^3) + 6c_1c_2 sinh(x^3)cosh(x^3) + 2c_2^2 + c_1^2 = 0$$
$$\frac{c_1^2}{(c_1x^3+c_2)^2} = 0$$
$$\sigma = 1 : 3[c_1^2 - c^2]sin^2(x^3) - 6c_1c_2 sin(x^3)cos(x^3) + 2c_2^2 - c_1^2 = 0$$

In each case this identity will only hold if $c_1 = c_2 = 0$; however, this will imply that $D_3\zeta_1 = 0$, which cannot happen. Thus the null killing vector ζ cannot be covariantly constant.

Case 2.2: Timelike Killing Vectors. If we require ζ to be timelike, equation (82) along with the fact that $\zeta_1 = e^A$ forces $\zeta_2^{(0)}$ to be greater than or equal to zero for all values of x^3. To find ζ_1 we integrate each of the three solutions to (78) given above

$$\sigma = -1 : \zeta_1 = \frac{sinh(x^3)}{c_1(c_1 cosh(x^3)+c_2 sinh(x^3))} + c_3 \tag{99}$$

$$\sigma = 0 : \zeta_1 = \frac{-1}{c_1(c_1x^3+c_2)} + c_3 \tag{100}$$

$$\sigma = 1 : \zeta_1 = \frac{sin(x^3)}{c_1(c_1 cos(x^3)+c_2 sin(x^3))} + c_3. \tag{101}$$

The inequality (80) restricts the choice of c_3 depending on the choice of c_1 and c_2:

$$\sigma = -1 : [c_1^2 + c_2^2]\zeta_1^2 - 2\left(\frac{c_1 sinh(x^3)+c_2 cosh(x^3)}{c_1 cosh(x^3)+c_2 sinh(x^3)}\right)\zeta_1 + \frac{1}{(c_1 cosh(x^3)+c_2 sinh(x^3))^2} < 0$$

$$\sigma = 0 : -c_1^2\zeta_1^2 - 2\left(\frac{c_1}{c_1x^3+c_2}\right)\zeta_1 + \frac{1}{(c_1x^3+c_2)^2} < 0 \tag{102}$$

$$\sigma = 1 : -[c_1^2 + c_2^2]\zeta_1^2 - 2\left(\frac{-c_1 sin(x^3)+c_2 cos(x^3)}{c_1 cos(x^3)+c_2 sin(x^3)}\right)\zeta_1 + \frac{1}{(c_1 cos(x^3)+c_2 sin(x^3))^2} < 0$$

Notice in both the null and timelike case, the value of σ restricts the domain of x^3. When $\sigma = 1$, the domain of x^3 is limited to a finite interval, $x^3 \in (x_0^3, x_0^3 + \pi)$, as the value $x_0^3 = arctan(-\frac{c_1}{c_2})$ will cause $(D_3\zeta_1)^{-\frac{1}{2}}$ to vanish. When $\sigma = 0$, $x^3 \geq -\frac{c_2}{c_1}$ to avoid singularities. In the case with $\sigma = -1$, $x^3 > x_0^3 = arctanh(-\frac{c_1}{c_2})$ when $c_1/c_2 \leq 1$, otherwise ζ_1 is regular on the whole of the real line.
Requiring ζ to be covariantly constant, equation (97) may be rewritten as a function set to zero in terms of ζ and $(D_3\zeta_1)^{\frac{1}{2}}$ for the three subcases with $\sigma = -1, 0, 1$ respectively:

$$[c_1 + 2c_1^2c_2c_3]sinh^2(x^3) + [c_2 + c_1^3c_3 + c_1c_2^2c_3]cosh(x^3)sinh(x^3) + [c_1^2c_2c_3 + c_1]$$
$$c_1c_3(c_1x^3 + c_2)$$
$$-[c_1 + 2c_1^2c_2c_3]sin^2(x^3) + [c_2 - c_1^3c_3 + c_1c_2^2c_3]cos(x^3)sin(x^3) + [c_1^2c_2c_3 + c_1]$$

In both cases where $\sigma = -1, 1$ the vanishing of the first and third equation will hold only if c_1 and c_2 both vanish, which violates the assumption $D_3\zeta_1 \neq 0$, and so there are no timelike covariantly constant vectors in either of these two cases. When $\sigma = 0$, setting the second equation to zero implies $c_3 = 0$, the Killing vector of the

form (89) with $\zeta_1 = -1/(c_1^2 x^3 + c_1 c_2)$ satsifies the condition in (97). A problem arises from the inequality (80)

$$-c_1^2\left(\frac{1}{c_1^2(c_1x^3+c_2)^2}\right) - 2\left(\frac{c_1}{(c_1x^3+c_2)}\right)\left(\frac{-1}{c_1(c_1x^3+c_2)}\right) + \frac{1}{(c_1x^3+c_2)^2} < 0;$$

simplifying the above leads to the inequality $2 < 0$ which is clearly impossible. We conclude there are no covariantly constant timelike vectors in the spacetimes belonging to Case 2.

Conclusions

To determine the subset of Kundt *CSI* spacetimes admitting a null or timelike isometry, several choices were made to simplify the Killing equations. Local coordinates were chosen so that one of the spacelike Killing vectors, Y, belonging to the (locally) homogeneous transverse space has been rectified to act locally as a translation in the x^3 direction, i.e., $Y = A\frac{\partial}{\partial x^3}$. The frame was then rotated so that the frame vector m^3 was aligned with the spatial part of ζ and, moreover, that the matrix m_{ie} was upper-triangular with $m_{33} = 1$. This causes the connection components Γ_{3ij} and Γ_{ij3} to vanish, simplifying the Killing equations considerably.

In this coordinate system we determined the special form for the components of ζ in terms of arbitrary functions and in terms of H and the W_e; i.e., (15) and (18). All of the functions involved (metric or otherwise) are expressed as polynomials in v with coefficient functions of u and x^e. These are substituted into the remaining Killing equations which are rearranged into the various orders of v to give (22) - (31), while the non-spacelike conditions yield (19) - (21). The highest order equation (22) gives two major subcases, either $D_3\zeta_1 = 0$ or $\Gamma^{(2)}_3 = 0$ in (17).

It is known that all *VSI* spacetimes admitting a non-spacelike isometry are *CCNV* spacetimes with ℓ as the covariantly constant vector [5]. As an analogue to this result, the equations arising from $\nabla_{[a}\zeta_{b]} = 0$ were examined to determine which *CSI* Kundt spacetimes admit a covariantly constant vector and which cannot.

The results of the analysis are summarized below:

Case 1.1.1: $\zeta = \ell$. In this case $R_{1212} = \sigma = 0$, the metric functions in (2) takes the form: $H(u, x^k)$ and $W_i(u, x^k)$. All *CSI* spacetimes in this subcase are clearly *CCNV* spacetimes with $\ell = \frac{\partial}{\partial v}$ covariantly constant

Case 1.1.2: $\zeta = \zeta_2(u, x^e)\ell$. With $R_{1212} = \sigma < 0$, the metric functions H and $W_i = m_i^e W_e$ will be of the form (52) while ζ_2 must satisfy the further constraint (53). These *CSI* spacetimes do not admit a covariantly constant vector.

Case 1.2.1a : $\zeta = n + \frac{\sigma^* v^2}{2}\ell$. The metric (2) with H and W_i take the form (66), R_{1212} may be any value in $\mathbb{R}$. There are no *CCNV* spacetimes belonging to this subset of *CSI* spacetimes.

Case 1.2.1b : $\zeta = n + \zeta_2(x^e)\ell$, $\zeta_2 \geq 0\ \forall x^e$. For any $\zeta_2^{(0)}(x^e) > 0$, $\forall x^e$, and any choice of $W_i^{(1)}(x^e)$ satisfying (50) with $\sigma = R_{1212} \leq 0$; the *CSI* Kundt spacetime with H and W_i given in (67) will admit a timelike Killing vector. If $\zeta_2^{(0)}(x^e) > 0$, $\forall x^e$, $\zeta = n$ will be a covariantly constant null vector.

If this Killing vector is covariantly constant, $W_i^{(1)} = 0$ and hence $\sigma = 0$, equation (42) and Lemma (0.1) imply $A_{ij} = 0$, and the remaining equations (40) and (41) force $\zeta_2^{(0)}$ to be constant. Thus ζ is the sum of the $CCNV$'s ℓ and n.

Case 1.2.2a : $\zeta = \ell + \zeta_2(u,x^e)n + \zeta_3(u)m^3$**.** For any ζ_3, and a particular choice of ζ_2 such that it satisfies the inequality $\zeta_3^2 \leq 2\zeta_2$ and the differential equation (69), the vector ζ will be a Killing vector for the CSI spacetime with metric functions given in (70) where $W_n^{(1)}$ and $W_n^{(0)}$ are, respectively, solutions (71) and (72). Due to the vanishing of $\sigma^* = 4\sigma + W_i^{(1)}W^{(1)i}$, the $W_i^{(1)}$ must also satisfy (50)

Requiring ζ to be a $CCNV$, the $W_i^{(1)}$ must vanish, causing $H = 0$ and $\sigma = 0$ as well; this is an example of a $CCNV$ metric with $\ell = \frac{\partial}{\partial v}$ as the $CCNV$ and ζ acting as a second $CCNV$. The additional constraints (39) - (42) require that $A_{ij} = 0$ along with the following simplification of ζ and the metric functions:

$$\zeta = n + \zeta_2\ell + m^3, \ \zeta_2 \in \mathbb{R}$$
$$H = 0, \ W_3(x^e) = w_3(x^e), \ W_n(x^e) = \int D_n(w_3)dx^3 + w_n(x^r),$$

If ζ is timelike, then $\zeta_2 > \frac{1}{2}$, while if ζ is null, $\zeta_2 = \frac{1}{2}$. All of the CSI spacetimes belonging to this subcase are automatically $CCNV$ with ℓ as another covariantly constant vector.

Case 1.2.2b : $n + [\frac{\sigma^* v^2}{2} + \zeta_2(u,x^e)]\ell + m^3$**.** For a particular choice of ζ_2 satisfying (73) the vector ζ is a Killing vector for the CSI spacetime with the metric functions in (74) where the $W_n^{(1)}$ and $W_n^{(0)}$ satisfy (75) and (76). The magnitude condition requires $\sigma^* > 0$ implying that $R_{1212} = \sigma > 0$. If ζ is now covariantly constant, a contradiction arises from (39), as it requires $D_1 H = \sigma v = 0$ despite the fact that we have assumed $\sigma \neq 0$. Thus the subset of CSI spacetimes associated with this subcase are never $CCNV$.

Case 2. Using a coordinate transform of type (5) with $g(u) = \frac{u}{\sqrt{|\sigma|}}$, σ in equation (10) is rescaled so that it equals $\sigma = -1, 0, 1$. Another coordinate transform of type (4) with $h = -\frac{\zeta_3^{(0)}}{D_3\zeta_1}$ causes all but one component to vanish:

$$W_3(u,x^3) = D_3 ln(D_3\zeta_1)v, \ W_n = 0 \quad .$$

The other Killing vector components may be expressed entirely in terms of ζ_1

$$\zeta_2 = \overbrace{(\frac{\sigma^*\zeta_1}{4} - D_3D_3\zeta_1)}^{\zeta_2^{(2)}}\frac{v^2}{2} + \overbrace{(H^{(1)}\zeta_1 - D_2\zeta_1)}^{\zeta_2^{(1)}}v + \zeta_2^{(0)}(u,x^e),$$
$$\zeta_3 = -D_3(\zeta_1)v.$$

Making one final coordinate transform of type (5) with $g = \int e^{-B(u)}du$ removes the u dependence from ζ_1 and, in fact, removes all u dependence from the other components of the Killing vector and the Killing equations, (i.e., (84) and (87)) involving $H^{(0)}$. Solving these yields the following algebraic equation for $H^{(0)}$

$$H^{(0)} = \frac{D_3(D_3(\zeta_1)\zeta_2^{(0)}) + D_3 D_3 ln((D_3\zeta_1)^{-\frac{1}{2}})\zeta_2^{(0)}\zeta_1}{\zeta_1^2 D_3 D_3 ln(\zeta_1 (D_3\zeta_1)^{-\frac{1}{2}})}$$

With all of the Killing equations satisfied, the non-spacelike conditions (82) and (80) give two subcases depending on whether ζ is a null or timelike Killing vector.

Case 2.1: $\zeta = \zeta_1 n + \left[\left(\frac{\sigma^*\zeta_1}{4} - D_3 D_3(\zeta_1)\right)\frac{v^2}{2}\right]\ell + [-D_3(\zeta_1)v]m^3$**.** If ζ is null, ζ_1 takes the following form, depending on the sign of σ:

$$\sigma = -1 : \zeta_1 = \frac{(c_1 cosh(x^3)+c_2 sinh(x^3))^{-1}}{c_1 sinh(x^3)+c_2 cosh(x^3) \pm \sqrt{c_1^2+c_2^2+(c_1 sinh(x^3)+c_2 cosh(x^3))^2}}$$

$$\sigma = 0 : \zeta_1 = \frac{1}{c_1(1\pm\sqrt{2})(c_1 x^3+c_2)}$$

$$\sigma = 1 : \zeta_1 = \frac{(c_1 cos(x^3)+c_2 sin(x^3))^{-1}}{-c_1 sin(x^3)+c_2 cos(x^3) \pm \sqrt{c_1^2+c_2^2+(-c_1 sin(x^3)+c_2 cos(x^3))^2}}$$

There are no covariantly constant null vectors in this subcase as the constraint in Lemma (0.1) on $W_3^{(1)}$ along with the identity (16) lead to a contradition with the given form of ζ_1.

Case 2.2: $\zeta = \zeta_1 n + \left[\left(\frac{\sigma^*\zeta_1}{4} - D_3 D_3(\zeta_1)\right)\frac{v^2}{2} + \zeta_2^{(0)}(x^3)\right]\ell + [-D_3(\zeta_1)v]m^3$**.** If ζ is to be timelike, depending on the sign of σ, ζ_1 takes the form:

$$\sigma = -1 : \zeta_1 = \frac{sinh(x^3)}{c_1(c_1 cosh(x^3)+c_2 sinh(x^3))} + c_3$$

$$\sigma = 0 : \zeta_1 = \frac{-1}{c_1(c_1 x^3+c_2)} + c_3$$

$$\sigma = 1 : \zeta_1 = \frac{sin(x^3)}{c_1(c_1 cos(x^3)+c_2 sin(x^3))} + c_3.$$

The inequality (80) restricts the choice of c_3 depending on the choice of c_1 and c_2:

$$\sigma = -1 : [c_1^2 + c_2^2]\zeta_1^2 - 2\left(\frac{c_1 sinh(x^3)+c_2 cosh(x^3)}{c_1 cosh(x^3)+c_2 sinh(x^3)}\right)\zeta_1 + \frac{1}{(c_1 cosh(x^3)+c_2 sinh(x^3))^2} < 0$$

$$\sigma = 0 : -c_1^2\zeta_1^2 - 2\left(\frac{c_1}{c_1 x^3+c_2}\right)\zeta_1 + \frac{1}{(c_1 x^3+c_2)^2} < 0$$

$$\sigma = 1 : -[c_1^2 + c_2^2]\zeta_1^2 - 2\left(\frac{-c_1 sin(x^3)+c_2 cos(x^3)}{c_1 cos(x^3)+c_2 sin(x^3)}\right)\zeta_1 + \frac{1}{(c_1 cos(x^3)+c_2 sin(x^3))^2} < 0$$

There are no timelike covariantly constant vectors in CSI spacetimes admitting ζ as a Killing vector.

Notice in both the null and timelike case, the value of σ restricts the domain of x^3. When $\sigma = 1$ the domain of x^3 limited to a finite interval, $x^3 \in (x_0^3, x_0^3 + \pi)$, as the value $x_0^3 = arctan(-\frac{c_1}{c_2})$ will cause $(D_3\zeta_1)^{-\frac{1}{2}}$ to vanish. When $\sigma = 0$, $x^3 \geq -\frac{c_2}{c_1}$ to avoid singularities. In the case with $\sigma = -1$ $x^3 > x_0^3 = arctanh(-\frac{c_1}{c_2})$ when $c_1/c_2 \leq 1$, otherwise ζ_1 is regular on the whole of the real line.

References

[1] A. Coley, R. Milson, V. Pravda and A. Pravdova, Class. Quant. Grav. **21**, 5519 (2004).
[2] V. Pravda, A. Pravdova, A. Coley and R. Milson,Class. Quant. Grav. **19**, 6213 (2002).
[3] A.Coley, A.Fuster, S.Hervik [gr-qc/0707.0957v2] [Add journal info if any].
[4] A. Coley, S. Hervik, G. Papadopoulos, N. Pelavas, Class. Quant. Grav. **26**, 105016 (2009) [gr-qc/0901.0394]
[5] A. Coley, A. Fuster, S. Hervik and N. Pelavas, JHEP 0705, 032 (2007).
[6] A. Coley, S. Hervik and N. Pelavas, Class. Quant. Grav. **23**, 3053 (2006). [gr-qc/0509113].
[7] A. Coley, S. Hervik and N. Pelavas, Class. Quant. Grav. **26**, 025013 (2009) [gr-qc/0901.0791]
[8] A. Coley, S. Hervik and N. Pelavas, Class. Quant. Grav. **26**, 125011 (2009) [gr-qc/0904.4877]
[9] A. Coley, R. Milson, V. Pravda and A. Pravdova, Class. Quant. Grav. **21**, L35 (2004).
[10] A. Coley, N. Pelavas, D. McNutt, IJGMMP **6** 3 (2009).
[11] V. Pravda, A. Pravdová, A. Coley and R. Milson, Class. Quant. Grav. **21**, 2873 (2004) [gr-qc/0401013].
[12] G.T. Horowitz and A.A. Tseytlin, Phys. Rev. D **51**, 2896 (1995).
[13] M. Caldarelli et al., Class. Quant. Grav. **24**, 1341 (2007).
[14] V.P. Frolov and A. Zelnikov, Phys. Rev. D **72**, 104005 (2005); V.P. Frolov and D.V. Fursaev, Phys. Rev. D **71**, 104034 (2005).
[15] H. Guggenheimer , 1977, Differential Geometry (Dover Publications, New York).

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

E-mail address: `aac, mcnuttd, pelavas@mathstat.dal.ca`